An overview on the equation $-\Delta u=u^p$ in bounded domains (Variational Problems and Related Topics)

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数理解析研究所講究録
第 1671 巻 2009 年 37-44

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1 Introduction

In this survey we consider the problem

$$\begin{cases}-\Delta u=u^{p} & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{1.1}$$ $\boxed{1}$

where Ω is a smooth bounded domain in $\mathbb{R}^{N}$, $N\geq 3$ and $p>1$. Problem (1.1) has been very studied in the last year and, despite of its simple structure, it is a great source of interesting phenomena and open problems. According to the values of the exponent p, we have the following classification: problem (1.1) is said

$$\textit{subcritical} \text{ if } 1<p<\frac{N+2}{N-2},$$
$$\textit{critical} \text{ if } p=\frac{N+2}{N-2},$$
$$\textit{supercritical} \text{ if } p>\frac{N+2}{N-2}.$$

In this survey we focus our interest mainly in the last case (Section 4). On the other hand, in order to explain the main difficulties, in Section 2 and Section 3 we list some of the most important results when $1<p\leq\frac{N+2}{N-2}$.

Some of the topics of this survey were treated in a lecture given by the author at the Kyoto University in June 2009. I would like to thank again all the organizers for their support and fantastic hospitality.

2 The subcritical case $1<p<\frac{N+2}{N-2}$

In this case it is not difficult to show that there exists at least one solution to (1.1) for any domain Ω. Indeed, if we consider the following minimization

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problem,

$$S_p = \inf_{\substack{u\in H_0^1(\Omega),\\ u\neq 0}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}} \tag{2.1}$$ 2

then, using the compactness of the imbedding of $H_0^1(\Omega)$ in $L^{p+1}(\Omega)$, it is easy to prove that S_p is achieved. This provides (up to a multiplicative constant), the existence of a solution to (1.1).
In next section we are going to see that this result in not true if $p = \frac{N+2}{N-2}$. For this reason it is interesting to study the asymptotic behavior of the solution which achieves S_p when $p \longrightarrow \frac{N+2}{N-2}$. We have the following result,

a3 **Theorem 2.1.** *(Han,[H], 1991) Let us suppose that u_ϵ is a solution to (1.1) which achieves (2.1) with $p = \frac{N+2}{N-2} - \epsilon$. Then, as $\epsilon \longrightarrow 0$,*

$$||u_\epsilon||_\infty \longrightarrow +\infty$$

$$\frac{u_\epsilon(x)}{\sqrt{\epsilon}} \longrightarrow C(p,N)G(x,x_0) \quad \textit{uniformly in } \Omega\setminus\{x_0\}$$

and x_0 verifies

$$\nabla R(x_0) = 0,$$

where $G(x,y)$ is the Green function of $-\Delta$ in $H_0^1(\Omega)$, $H(x,y) = \frac{1}{N(N-2)\omega_N} - G(x,y)$ is its regular part and $R(x) = H(x,x)$. Here $C(p,N)$ is a positive real constant depending only on p and N.

Solutions verifying $||u_\epsilon||_\infty \longrightarrow +\infty$ at one point and $u_\epsilon(x) \longrightarrow 0$ far away from its maximum point are usually called *single − bump* solutions. Han's result claims that the solution founded minimizing (2.1) is a single-bump solution as $p \longrightarrow \frac{N+2}{N-2}$.
In an analogous way we define *k − bump* solutions if the same behavior occurs at k points.

3 The critical case $p = \frac{N+2}{N-2}$

It is virtually impossible to provide a complete list of results in the critical case. We just mention some of our interest. First, the existence result of the previous section is not true anymore if we consider the *critical* or the *supercritical* case. Indeed, we have the following fundamental result:

3 **Theorem 3.1.** *(Pohozaev,[P], 1965) Let us suppose that Ω is starshaped with respect to some point. Then there is no solution to (1.1) for $p \geq \frac{N+2}{N-2}$.*

The proof of Theorem 3.1 is a consequence of the celebrated *Pohozaev identity* which dealt with solutions of the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$ 4

So if u solves (3.1) we have that

$$\frac{2-N}{2}\int_{\Omega} uf(u) + N\int_{\Omega} F(u) = \frac{1}{2}\int_{\partial\Omega}(x-y)\cdot\nu\left(\frac{\partial u}{\partial\nu}\right)^2 \quad \textit{(Pohozaev identity)}$$

where $F(s)=\int_0^s f(t)dt$ and ν is the outer normal at $\partial\Omega$. If we use the Pohozaev identity with $f(s)=s^p$, $p\geq\frac{N+2}{N-2}$, we get that in star-shaped domains necessarily $u\equiv 0$. Theorem 3.1 emphasizes the role of the geometry of the domain in order to derive nonexistence result in the critical case. Note that if Ω is not star-shaped with respect to any point but it has, for example, a nontrivial topology, the situation is completely different. This is showed in the following example,

6 **Theorem 3.2.** *(Kazdan and Warner,[KW],* 1975*) Let Ω be an annulus. Then there exists a radial solution to (1.1) for any $p>1$.*

The proof of the previous theorem is very similar to one of the subcritical case. Indeed, let us consider the following infimum,

$$S_{rad} = \inf_{\substack{u\in H_{rad}(\Omega),\\ u\not\equiv 0}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}} \tag{3.2}$$ 7

where $H_{rad}(\Omega)=\{u\in H_0^1(\Omega): u(x)=u(|x|)\}$. Since the imbedding of $H_{rad}(\Omega)$ in $L^{p+1}(\Omega)$ is compact for any $p>1$ we derive the existence of a solution.
On the other hand the non-spherical case is much harder to handle. An important contribution was given by Coron,

8 **Theorem 3.3.** *(Coron,[C],* 1984*) Let Ω be a domain with a "small hole". Then there exists a solution to (1.1) for $p=\frac{N+2}{N-2}$.*

This theorem was extended some years later by Bahri and Coron, which prove this beautiful (and deep!) result.

9 **Theorem 3.4.** *(Bahri and Coron,[BC],* 1988*) If there exists a positive integer d such that $H_d(\Omega,\mathbb{Z}_2)\neq 0$, then there exists a solution to (1.1) for $p=\frac{N+2}{N-2}$.*

Here $H_d(\Omega,\mathbb{Z}_2)\neq 0$, denotes the homology of dimension d with $\mathbb{Z}_2$ coefficients. In particular, if $N=3$, Bahri and Coron's results implies that if Ω is not contractible there exists a solution to (1.1).
Now we mention the most important paper regarding the critical case: the pioneering paper by Brezis and Nirenberg. In order to handle the obstruction given by the Pohozaev identity, they added a linear term to the equation and obtained the following beautiful result:

b10 **Theorem 3.5.** *(Brezis and Nirenberg,[BN],* 1983*) Let us consider the problem*

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}}+\lambda u & \textit{in } \Omega,\\ u>0 & \textit{in } \Omega\\ u=0 & \textit{on } \partial\Omega, \end{cases} \tag{3.3}$$ b11

Then there exists $\lambda^* \geq 0$ such that for any $\lambda^* < \lambda < \lambda_1$ there exists one solution to (3.3). Moreover we have that $\lambda^* = 0$ if $N \geq 4$. Here λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Note that, using again the Pohozaev identity, if $\lambda \leq 0$ in (3.3), there is no solution in star-shaped domains, so the Brezis and Nirenberg's result is sharp.

We end this section on the critical case by mentioning some interesting examples due to Dancer ([D], 1988), Ding ([Di], 1988) and Passaseo ([Pa], 1989). Here the authors perturb some contractible domains in order to derive an existence result to (1.1) in non-contractible domains.

It worths to observe that the results of this section rely on the fundamental remark that it is possible to associate to the problem (1.1) a limit problem given by

$$-\Delta u = u^{\frac{N+2}{N-2}} \text{ in } \mathbb{R}^N, \qquad (3.4)$$ b12

whose solutions are completely classified (see [CGS]).

4 The supercritical case $p > \frac{N+2}{N-2}$

This case is much more difficult to manage since there is no imbedding of $H_0^1(\Omega)$ in $L^{p+1}(\Omega)$. For this reason, standard variational methods does not apply directly.

Let us start this section by considering the case where the exponent p is slightly grater than the critical one, namely $p = \frac{N+2}{N-2} + \epsilon$. We have the following result:

c1 **Theorem 4.1.** *(Ben Ayed, El Mehdi, Grossi and Rey,[BEGR], 2003) Let us consider the problem (1.1) with $p = \frac{N+2}{N-2} + \epsilon$. Then, for any domain Ω, there is not any single-bump solution for ϵ small.*

We recall that if $p = \frac{N+2}{N-2} - \epsilon$ (subcritical case) there always exists one solution to (1.1). From the last result we see that it is not allowed to exchange ϵ with $-\epsilon$! On the other hand, if we look for solutions with a large number of bumps, Theorem 4.1 is not true anymore. Indeed we have,

c2 **Theorem 4.2.** *Let us consider the problem (1.1) with $p = \frac{N+2}{N-2} + \epsilon$. We have that,*
1) If Ω is a domain with one hole then, for ϵ small enough, there exists a $2-$bumps solution (del Pino, Felmer and Musso, [DFM], 2003),
2) If Ω is a domain with one hole then, for ϵ small enough, there exists a $3-$bumps solution (Pistoia and Rey, [PR], 2006).

These results lead naturally to the following

Open problems Let $p = \frac{N+2}{N-2} + \epsilon$ in (1.1).
(1) If Ω is a domain with one hole, does exists, for ϵ small enough, a $k-$*bumps* solution for any $k \geq 4$?
(2) Does exist any domain Ω such that there are **no** $2-$*bumps* solutions?

The latest theorems concerned with (supercritical) perturbation of the critical case. Next results deal with exponent p "far" from the critical one. The first one is

c3 **Theorem 4.3.** *(Passaseo, [Pa2],* 1992*) There exists a contractible domain such that for any $p \geq \frac{N+2}{N-2}$ there exists a solution to (1.1).*

We also have the following interesting nonexistence result,

c4 **Theorem 4.4.** *(Passaseo, [Pa3],* 1993*) There exists a domain with nontrivial topology such that for any $p > \frac{N+1}{N-3}$ there exist no solution to (1.1).*

Note that the exponent appearing in Passaseo's theorem is the critical Sobolev exponent in dimension $N-1$. This result is somehow surprising: unlike to the critical case, the topology of the domain is not a sufficient condition for the existence of solutions! Moreover this result is sharp, as follows by the next theorem:

c5 **Theorem 4.5.** *(del Pino, Musso and Pacard, [DMP],* 2009*) Let us consider the same domain of Theorem 4.4. Then, for ϵ small enough, there exists a solution to (1.1) with $p = \frac{N+1}{N-3} - \epsilon$. Moreover, as $\epsilon \to 0$, the solution concentrates along a curve.*

On the other hand, if the domain has a small hole, the topology of the domain ensures the existence of solutions. This is a generalization of Coron's result to the supercritical case,

c6 **Theorem 4.6.** *(del Pino and Wei, [DW],* 2007*) Let Ω be a domain with a circular hole. Then there exists a sequence of exponents $p_1 < p_2 < ..$ with $\lim_{k \to +\infty} p_k = +\infty$ such that if $p \neq p_k$ there is a solution to (1.1) provided the hole is small enough.*

We now consider a different type of results, i.e. we look for solutions to (1.1) when p is large. This approach is justified by the existence of a limit problem to (1.1) as $p \to +\infty$. This was done in [G], where the author studied the radial solution in the annulus founded by Kazdan and Warner in Theorem 3.2. We have the following result,

c7 **Theorem 4.7.** *(Grossi, [G],* 2006*) Let u_p the unique radial solution of (1.1). Then as $p \to +\infty$*

$$u_p(|x|) \to w(|x|) \quad \text{in } C^0(\bar{A}) \tag{4.1}$$ c8

with

$$w(|x|) = \frac{2}{a^{2-N} - b^{2-N}} \begin{cases} a^{2-N} - |x|^{2-N} & \text{for } a \leq |x| \leq r_0 \\ |x|^{2-N} - b^{2-N} & \text{for } r_0 \leq |x| \leq b \end{cases}$$

where

$$r_0 = \left(\frac{a^{2-N} + b^{2-N}}{2} \right)^{\frac{1}{2-N}}.$$

Unlike to the case of single-bump solution, we have that no concentration occurs as $p \to +\infty$. We point out that, if we denote by $G(r,s)$ the Green function of the operator $-u'' - \frac{N-1}{r}u'$ in $H_0^1(a,b)$ and by $H(r,s)$ its regular part we have that

$$w(|x|) = \frac{G(|x|, r_0)}{H(r_0, r_0)}$$

Theorem 4.7 is the starting point to deduce some existence result to (1.1) when p is large. Together with (4.1) we also need to derive a limit problem to (1.1) for p large. This can be done setting

$$\tilde{u}_p(r) = \frac{p}{\|u_p\|_\infty}\left(u_p(\epsilon_p r + r_p) - \|u_p\|_\infty\right), \tag{4.2}$$ c9

where $u_p(r_p) = \|u_p\|_\infty$ and $p\epsilon_p^2\|u_p\|_\infty^{p-1} = 1$. We have that

$$\tilde{u}_p \to U \quad \text{in } C^1_{loc}(\mathbb{R}), \tag{4.3}$$ c10

where

$$U(r) = \log\frac{4e^{\sqrt{2}r}}{(1+e^{\sqrt{2}r})^2} \tag{4.4}$$ c11

is the only solution of the problem:

$$\begin{cases} -z'' = e^z & \text{in } \mathbb{R} \\ z(0) = z'(0) = 0. \end{cases} \tag{4.5}$$ c12

Using these information we can try to construct a radial solution in Brezis-Nirenberg type problem, i.e. adding a linear term to the equation. It was done in the unit ball B_1 (for p large). We have that

c13 **Theorem 4.8.** *(Grossi, [G1], 2008) Let us consider the problem*

$$\begin{cases} -\Delta u + a(|x|)u = u^p & \text{in } B_1, \\ u > 0 & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{4.6}$$ c14

and let us denote by $G_a(r,s)$ the Green function of the operator $-u'' - \frac{N-1}{r}u' + a(r)u$ in $H_0^1(0,1)$ and by $H_a(r,s)$ its regular part. Then, if r_1 is a nondegenerate critical point of $H_a(r,r)$, for p large enough there exists a radial solution u_p to (4.6). Moreover we have that

$$u_p(|x|) \to \frac{G_a(|x|, r_1)}{H_a(r_1, r_1)}.$$

This result holds for radial solutions in the unit ball and it is not easy to extend it to a non-spherical situation. However, coming back to problem (1.1), we have the following open problem,

Open problem Let Ω be a domain with one hole (not necessarily small). Then, for p large enough, does there exist a solution to (1.1) which satisfies:

i) $u_p \to 1$ in M for $p \to +\infty$
ii) $\Delta u_p \to 0$ outside of M for $p \to +\infty$?

Observe that this solution should be the "natural" extension of the one in Theorem (4.7) to non-spherical domain.

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